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MATHEMATICS
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METHOD OF DIFFERENTIATION

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions, derivatives of implicit functions, derivatives up to order two, L'Hospital rule of evaluation of limits of functions, geometrical interpretation of the derivative

A. DERIVATIVE USING FIRST PRINCIPLE / AB INITIO METHOD

If f(x) is a derivable function then, $\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Limit}} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$

Ex.1 Using first principles, find the derivative of the function $y = -\cot x - x$.
We find $\Delta y = -\cot(x + \Delta x) - (x + \Delta x) + \cot x + x = \cot x - \cot(x + \Delta x) - \Delta x$.

Sol. Using the formula $\cot\alpha - \cot\beta = \frac{\sin(\beta - \alpha)}{\sin\alpha \sin\beta}$, we get $\Delta y = \frac{\sin(x + \Delta x - x)}{\sin x \sin(x + \Delta x)} - \Delta x = \frac{\sin\Delta x}{\sin x \sin(x + \Delta x)} - \Delta x$,

whence $\frac{\Delta y}{\Delta x} = \frac{\frac{\sin\Delta x}{\Delta x}}{\sin x \sin(x + \Delta x)} - 1$ & consequently $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\frac{\sin\Delta x}{\Delta x}}{\sin x \cdot \sin(x + \Delta x)} - 1 = \frac{1}{\sin^2 x} - 1$.

Thus we have $y' = \frac{1}{\sin^2 x} - 1 = \cot^2 x$.

Ex.2 Find by first principle the derivative of $\frac{e^x}{x}$ w.r.t. x.

Sol. $y = \frac{e^x}{x}$; $y + \Delta y = \frac{e^{x+\Delta x}}{x+\Delta x} \Rightarrow \Delta y = \frac{e^{x+\Delta x}}{x+\Delta x} - \frac{e^x}{x} = \frac{x.e^x.e^{\Delta x} - e^x(x+\Delta x)}{x(x+\Delta x)}$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{xe^x \frac{\left(e^{\Delta x} - 1\right)}{\Delta x} - e^x \cdot \frac{\Delta x}{\Delta x}}{x(x+\Delta x)} \Rightarrow \underset{\Delta x \to 0}{\text{Limit}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Limit}} \left[\frac{xe^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) - e^x}{x(x+\Delta x)}\right]$$

$$\Rightarrow \frac{d}{dx}(x\, e^x) = \frac{xe^x - e^x}{x^2}$$

Ex.3 If $f(x) = (\ln x)^x$, find $f'(x)$ from the first principle.

Sol. By definition $f'(x) = \underset{h \to 0}{\text{Limit}} \frac{[\ell n(x+h)]^{x+h} - (\ell n x)^x}{h}$

$$= \underset{h \to 0}{\text{Limit}} \frac{e^{(x+h)\ell n[\ell n(x+h)]} - e^{x\ell n(\ell n x)}}{h} = e^{x \ln(\ln x)} \underset{h \to 0}{\text{Limit}} \left[\frac{e^{(x+h)\ell n(\ell n(x+h)) - x\ell n(\ell n x)} - 1}{h}\right]$$

$$= (\ln x)^x \underset{\substack{h \to 0 \\ s \to 0}}{\text{Limit}} \left[\frac{e^s - 1}{s} \cdot \frac{s}{h}\right]$$ where $s = (x + h) \ln[\ln(x + h)] - x \ln(\ln x)$

Note that as $h \to 0$, $s \to 0 \Rightarrow \underset{s \to 0}{\text{Limit}} \frac{e^s - 1}{s} = 1$

$$= (\ln x)^x \underset{h \to 0}{\text{Limit}} \frac{(x+h)\ell n[\ell n(x+h)] - x\ell n(\ell n x)}{h}$$

$$= (\ln x)^x \left[x\left\{\lim_{h\to 0}\frac{\ln(\ln(x+h)) - \ln(\ln x)}{h}\right\} + \lim_{h\to 0}\ln(\ln(x+h))\right] \quad ....(1)$$

l say

Now $$l = \lim_{h\to 0}\frac{\ln\left(\frac{\ln(x+h)}{\ln x}\right)}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{\ln x\left(1+\frac{h}{x}\right)}{\ln x}\right)}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{\ln x + \ln\left(1+\frac{h}{x}\right)}{\ln x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{\ln\left(1+\frac{h}{x}\right)}{\ln x}\right)}{h} = \lim_{h\to 0} \ln\left(1+\frac{\ln\left(1+\frac{h}{x}\right)}{\ln x}\right)^{1/h} = \ln e^{\lim_{h\to 0}\frac{1}{h}\left(\frac{\ln\left(1+\frac{h}{x}\right)}{\ln x}\right)}$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)^{1/h}}{\ln x} = \frac{\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}}{\ln x} \Rightarrow l = \frac{1}{x\ln x}$$

Substituting the value of l in (1)

$$= (\ln x)^x \cdot \left[\left\{x\left(\frac{1}{x\ln x}\right)\right\} + \ln(\ln x)\right] = (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

Derivative Of Standard Functions :

(i) $D(x^n) = n.x^{n-1}$; $x \in R$, $n \in R$, $x > 0$

(ii) $D(e^x) = e^x$

(iii) $D(a^x) = a^x . \ln a \quad a > 0$

(iv) $D(\ln x) = \frac{1}{x}$

(v) $D(\log_a x) = \frac{1}{x}\log_a e$

(vi) $D(\sin x) = \cos x$

(vii) $D(\cos x) = -\sin x$

(viii) $D(\tan x) = \sec^2 x$

(ix) $D(\sec x) = \sec x . \tan x$

(x) $D(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$

(xi) $D(\cot x) = -\operatorname{cosec}^2 x$

(xii) $D(\text{constant}) = 0$ where $D = \frac{d}{dx}$

(xiii) $D(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1<x<1$

(xiv) $D(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1<x<1$

(xv) $D(\tan^{-1} x) = \frac{1}{1+x^2}, x\in R$

(xvi) $D(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x|>1$

(xvii) $D(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x|>1$

(xviii) $D(\cot^{-1} x) = \frac{-1}{1+x^2}, x\in R$

B. RULES OF DIFFERENTIATION

If u and v are derivable function of x, then ,

(i) $\frac{d}{dx}(u\pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(Ku) = K\frac{du}{dx}$, where K is any constant

The extended linearity rule

If $f_1, f_2, ..., f_n$ are differentiable functions and $a_1, a_2, ..., a_n$ are constants, then

$$\frac{d}{dx}[a_1f_1 + a_2f_2 + ... + a_n f_n] = a_1\frac{df_1}{dx} + a_2\frac{df_2}{dx} + ... + a_n\frac{df_n}{dx}$$

(iii) PRODUCT RULE : $\frac{d}{dx}(u\,.\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$ i.e. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

On division by uv the above result may be written as $\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}$.

Hence it is clear that the rule may be extended to product of more than two functions.
For example, if y = uvw; let vu = z, then y = uz.

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{z}\frac{dz}{dx}$ but $\frac{1}{z}\frac{dz}{dx} = \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$,

by substitution $\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$.

Generally, if y = uvwt.....

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \frac{1}{t}\frac{dt}{dx} + \ldots\ldots,$$

and if we multiply be uvwt..... we obtain $\frac{dy}{dx}$ = (vwt ...) $\frac{du}{dx}$ + (uwt...) $\frac{dv}{dx}$ + (uvt...) $\frac{dw}{dx}$ + ...,

i.e. multiply the differential coefficient of each separate function by the product of all the remaining functions and add up all the results; the sum will be the differential coefficient of the product of all the functions.

(iv) QUOTIENT RULE : $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$ where $v \neq 0$

The derivation of formula for the derivative of the quotient of two functions f and g as follows :

$$\frac{d}{dx}\left(\frac{f}{g}\right)(a) = \lim_{x\to a}\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a}\frac{1}{g(x)g(a)}.\frac{f(x)g(a) - f(a)g(x)}{x-a}$$

$$= \frac{1}{g^2(a)}\lim_{x\to a}\frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{x-a}$$

$$= \frac{1}{g^2(a)}\left[\lim_{x\to a} g(a)\frac{f(x)-f(a)}{x-a} - \lim_{x\to a} f(a)\frac{g(x)-g(a)}{x-a}\right] = \frac{f(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

(v) CHAIN RULE : If f and g are differentiable functions, then so is the composite function f(g(x)). Let a be a number in the domain of g such that g(a) is in the domain of f.

$$[f(g)]' = \lim_{x\to a}\frac{(f(g))(x) - (f(g))(a)}{x-a} = \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x-a}.$$

$$[f(g)]'(a) = \lim_{x\to a}\left[\frac{f(g(x)) - (f(g(a))}{g(x)-g(a)}\,\frac{g(x)-g(a)}{x-a}\right] = \left[\lim_{x\to a}\frac{f(g(x)) - (f(g(a))}{g(x)-g(a)}\right]\left[\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right].$$

Setting y = g(x) and b = g(a) and noting that y approaches b as x approaches a, we have

$$[f(g)]'(a) = \lim_{y\to b}\frac{f(y)-f(b)}{y-b}\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(b)\,g'(a)$$

$$= f'(g(a))\,g'(a) = (f'(g)g')(a)$$

Remark : In using the Chain Rule we work from the outside to inside. Formula says that we differentiate the outer function f[at the inner function f(x)] and then we multiply by the derivative of the inner function.

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}=\underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\cdot\underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Ex.4 Differentiate $y = \sin(x^2)$

Sol. If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx}=\frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}=\underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\cdot\underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Ex.5 If $F(x) = (x^2 + 2)^3$, compute $F'(x)$. One way to do this problem is to expand $(x^2 + 2)^3$ and use the differentiation formulae.

Sol. $F(x) = (x^2 + 2)^3 = x^6 + 6x^4 + 12x^2 + 8$,
$F'(x) = 6x^5 + 24x^3 + 24x$.
Another method uses the Chain Rule. Let g and f be the functions defined, respectively, by $g(x) = x^2 + 2$ and $f(y) = y^3$. Then $f(g(x)) = (x^2 + 2)^3 = F(x)$, and, according to the Chain Rule,

$$F'(x) = [f(g(x))]' = f'(g(x))\, g'(x).$$

Since $g'(x) = 2x$ and $f'(y) = 3y^2$, we get $f'(g(x)) = 3(x^2 + 2)^2$ and

$$F'(x) = 3(x^2 + 2)^2 (2x) = 6x (x^4 + 4x^2 + 4),$$

which agrees with the alternative solution above.

If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ **" Chain Rule "**

Let f and g be two differentiable functions. The formation of the composite function f(g) is suggested by writing $u = g(x)$ and $y = f(u)$. Thus x is transformed by g into u, and the resulting u is then transformed by f into $y = f(u) = f(g(x))$. We have

$$\frac{du}{dx} = g'(x), \qquad \frac{dy}{du} = f'(x) \qquad \frac{dy}{dx} = [f(g(x))]'$$

By the Chain Rule, $[f(g(x))]' = f'(g(x))\, g'(x) = f'(u)g'(x)$, and so $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$. ...(1)

The idea that one can simply cancel out du in (1) is very appealing and accounts for the popularity of the notation. It is important to realize that the cancellation is valid because the Chain Rule is true, and not vice versa. Thus, du is simply a part of the notation for the derivative and means nothing by itself. Note also that (1) is incomplete in the sense that it does not say explicitly at what points to evaluate the derivatives. We can add this information by writing

$$\frac{dy}{dx}(a)=\frac{dy}{du}(a)\frac{du}{dx}(a).$$

Ex.6 If $w = z^2 + 2z + 3$ and $z = \frac{1}{x}$, find $\frac{dw}{dx}(2)$

Sol. By the Chain Rule, $\frac{dw}{dx}=\frac{dw}{dz}\frac{dz}{dx} = (2z + 2)\left(-\frac{1}{x^2}\right)$.

when $x = 2$, we have $z = \frac{1}{2}$. Hence $\frac{dw}{dx}(2) = \left(2.\frac{1}{2}+2\right)\left(-\frac{1}{4}\right)=-\frac{3}{4}$.

Ex.7 Differentiate $g(x) = \sqrt[4]{\frac{x}{1-3x}}$.

Sol. Write $g(x) = \left(\frac{x}{1-3x}\right)^{1/4} = u^{1/4}$ where $u = \frac{x}{1-3x}$ is the inner function and $u^{1/4}$ is the outer function.

Then, $g'(x) = (u^{1/4})'u'(x) = \frac{1}{4}u^{-3/4}u'(x)$ and we have $g'(x) = \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left(\frac{x}{1-3x}\right)'$

$$= \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{(1-3x)(1)-x(-3)}{(1-3x)^2}\right] \qquad = \frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{1}{(1-3x)^2}\right] = \frac{1}{4x^{3/4}(1-3x)^{5/4}}.$$

Ex.8 Differentiate $g(x) = \cos x^2 + 5\left(\frac{3}{x}+4\right)^6$.

Sol. $\frac{dg}{dx} = \frac{d}{dx}\cos x^2 + 5\frac{d}{dx}(3x^{-1}+4)^6 = -\sin x^2\frac{d}{dx}(x^2) + 5\left[6(3x^{-1}+4)^5\frac{d}{dx}(3x^{-1}+4)\right]$

$= (-\sin x^2)(2x) + 30(3x^{-1}+4)^5(-3x^{-2}) \qquad = -2x\sin x^2 - 90x^{-2}(3x^{-1}+4)^5$

Ex.9 Differentiate $y = \cos^4(3x+1)^2$.

Sol. $\frac{dy}{dx} = 4\cos^3(3x+1)^2\frac{d}{dx}\cos(3x+1)^2 = 4\cos^3(3x+1)^2 . [-\sin(3x+1)^2] . \frac{d}{dx}(3x+1)^2$

$= -4\cos^3(3x+1)^2\sin(3x+1)^2 . 2(3x+1)(3) = -24(3x+1)\cos^3(3x+1)^2\sin(3x+1)^2.$

Ex.10 If $y = \sin\sqrt{\cos x}$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{d\sin\sqrt{\cos x}}{dx} = \frac{d\sin\sqrt{\cos x}}{d\sqrt{\cos x}} . \frac{d\sqrt{\cos x}}{d\cos x} . \frac{d(\cos x)}{dx}$

$= \cos\sqrt{\cos x} . \frac{1}{2\sqrt{\cos x}} . (-\sin x) = -\frac{\sin x\cos\sqrt{\cos x}}{2\sqrt{\cos x}}$

Ex.11 If $y = x^2\cos(\log x)$, find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\{x^2\cos(\log x)\} = x^2\frac{d\{\cos(\log x)\}}{dx} + \cos(\log x)\frac{d(x^2)}{dx}$

$= x^2 . \frac{d\cos(\log x)}{d\log x}\cdot\frac{d(\log x)}{dx} + \cos(\log x) . 2x$

$= x^2(-\sin(\log x)\} . \frac{1}{x} + 2x\cos(\log x) \qquad = -x\sin(\log x) + 2x\cos(\log x)$

C. LOGARITHMIC DIFFERENTIATION

To find the derivative of :

(i) a function which is the product or quotient of a number of functions **OR**

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate. This is called **Logarithmic Differentiation**.

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation y = f(x) and use the Laws of Logarithms to simplify.

2. Differentiate both sides with respect to x.

3. Solve the resulting equation for y′.

If f(x) < 0 for some values of x, then in f(x) is not defined, but we can write |y| = |f(x)| and use $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$.

Ex.12 Find **(a)** $\frac{d}{dx} \ln|x|$ **(b)** $\frac{d}{dx} \ln |x^2 - x|$

Sol. **(a)** For all $x \neq 0$, $\frac{d}{dx} \ln|x| = \frac{1}{x}\frac{dx}{dx} = \frac{1}{x}$.

In contrast, $\frac{d}{dx} \ln x = \frac{1}{x}$ but only for x > 0.

(b) $\frac{d}{dx} \ln |x^2 - x| = \frac{1}{(x^2 - x)}\frac{d}{dx}(x^2 - x) = \frac{2x-1}{x(x-1)}$, $x \neq 0$, $x \neq 1$.

In contrast, $\frac{d}{dx}(\ln (x^2 - x) = \frac{2x-1}{x(x-1)}$ but only if x (x − 1) > 0, that is, for x > 1 or x < 0.

Ex.13 Differentiate $y = x^{\sqrt{x}}$.

Sol. Using logarithmic differentiation, we have $\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$

$$\frac{y'}{y} = \sqrt{x}.\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}} \qquad y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

Ex.14 Find $\frac{dy}{dx}$, where $y = (x + 1)^{2x}$.

Sol. $y = (x + 1)^{2x} \Rightarrow \ln y = \ln[(x + 1)^{2x}] = 2x \ln(x + 1)$

Differentiate both sides of this equation : $\frac{1}{y}\frac{dy}{dx} = 2x\left\{\frac{d}{dx}[\ln(x+1)]\right\} + \left[\frac{d}{dx}(2x)\right] \ln (x + 1)$

$$= 2x\left[\frac{1}{x+1}(1)\right] + 2 \ln (x + 1) = \frac{2x}{x+1} + 2 \ln (x + 1)$$

Finally, multiply both sides by $y = (x + 1)^{2x}$: $\frac{dy}{dx} = \left[\frac{2}{x+1} + 2\ln(x+1)\right](x + 1)^{2x}$

If u and v be both functions of x, it appears that the general formula

$$\frac{dy}{dx} = u^v \log_e u \frac{dv}{dx} + vu^{v-1}\frac{du}{dx}$$

is the sum of the two special forms and therefore we may, instead of taking logarithms in any particular example, consider first u constant and then v constant and add the results obtained on these suppositions.

Ex.15 If $y = e^x \sin x^3 + (\tan x)^x$, find $\frac{dy}{dx}$

Sol. Let $u = e^x \sin x^3$ and $v = (\tan x)^x$. Now $u = e^x \sin x^3$

Differentiating w. r. t. x, we get $\frac{du}{dx} = e^x . \frac{d\{\sin(x)^3\}}{dx} + \sin x^3 . \frac{d}{dx}(e^x) = e^x . \cos x^3 . 3x^2 + \sin x^3 . e^x$

Hence $\frac{du}{dx} = 3x^2 e^x \cos x^3 + e^x \sin x^3$ and $v = (\tan x)^x$ $\therefore$ $\log v = x \log(\tan x)$

Differentiating w. r. t. x, we get $\frac{1}{v}\frac{dv}{dx} = 1. \log(\tan x) + x . \frac{1}{\tan x} \sec^2 x$

$$\therefore \quad \frac{dv}{dx} = v[\log(\tan x) + x \cot x . \sec^2 x] = (\tan x)^x [\log(\tan x) + x \cot x \sec^2 x]$$

Now $y = u + v$ $\therefore$ $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = 3x^2 e^x \cos(x^3) + e^x \sin(x^3) + (\tan x)^x [\log(\tan x) + x\cot x \sec^2 x]$

D. PARAMETRIC DIFFERENTIATION

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Derivative Of A Function w.r.t. Another Function

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

E. DIFFERENTIATION OF IMPLICIT FUNCTIONS

Assume that the equation F(x, y) = 0 specifies y as an implicit function of x. In what follows we shall consider this function to be differentiable.

Differentiating both sides of the equation F(x, y) = 0 with respect to x, we obtain a first-degree equation with respect to y′. This equation easily yields y′, that is, the derivative of the implicit function.

Ex.16 Find $\frac{dy}{dx}$ from the equation $x^3 + \ln y - x^2 e^y = 0$.

Sol. Differentiating both sides of the equation with respect to x, we obtain

$$3x^2 + \frac{y'}{y} - x^2 e^y y' - 2xe^y = 0, \quad \text{i.e.} \quad y' = \frac{(2xye^y - 3x^2)y}{1 - x^2 ye^y}.$$

Steps for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx by dividing both sides of the equation by the left-hand factor that does not contain dy/dx.

Ex.17 Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Sol. **1.** Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation.

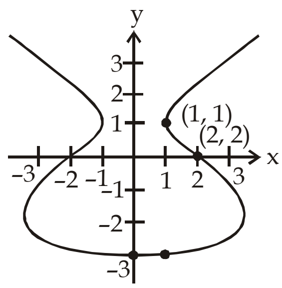
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$$



$y^3 + y^2 - 5y - x^2 = -4$

Figure

The derivative is $\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$

Point on Graph	Slope of Graph
(2, 0)	$-\frac{4}{5}$
(1, –3)	$\frac{1}{8}$
x = 0	0
(1, 1)	Undefined

Ex.18 Find $\frac{dy}{dx}$ if $xy - x = 1$.

Sol. We must find $\frac{dy}{dx}$ at x = 1. Assume y is a function of x, y = y(x). The relation now is $xy(x) - x = 1$.

Hence, $\frac{d}{dx}[xy(x) - x] = \frac{d}{dx}(1)$ $\quad \frac{d}{dx}[xy(x)] - \frac{d}{dx}(x) = 0.$ ...(1)

By the product rule, $\frac{d}{dx}[x \,.\, y(x)] = x\frac{d}{dx}[y(x)] + \left(\frac{d}{dx}(x)\right)y(x) = x\frac{dy}{dx} + y(x).$

Hence, substituting into (1) we obtain $x\frac{dy}{dx} + y(x) - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1-y(x)}{x}$.

Remark If, in the preceding example, we had explicitly solved for y, we would have obtained

$$y = 1 + \frac{1}{x} \qquad \frac{dy}{dx} = -\frac{1}{x^2}.$$

To see that this is exactly the result obtained by implicit methods, note that if 1 + 1/x is substituted for y $\Rightarrow \frac{dy}{dx} = \frac{1-y(x)}{x} = \frac{1-[1+(1/x)]}{x} = -\frac{1}{x^2}$.

Ex.19 Find dy/dx if $2x^2 + xy - 3y^2 = x$.

Sol. If we assume y is a function of x and represent it by y(x), the equation reads $2x^2 + xy(x) - 3[y(x)]^2 = x$.

Hence, $\frac{d}{dx}[2x^2 + xy(x) - 3[y(x)]^2] = \frac{d}{dx}(x)$.

Using the sum rule, we obtain $\frac{d}{dx}(2x^2) + \frac{d}{dx}[xy(x)] - 3\frac{d}{dx}[y(x)]^2 = 1$. ...(1)

By the product rule, $\frac{d}{dx}[xy(x)] = x\frac{dy}{dx} + y\frac{d}{dx}(x) = x\frac{dy}{dx} + y \Rightarrow \frac{d}{dx}[y(x)]^2 = 2y(x)\frac{dy}{dx}$.

Substituting these results into Formula (1), we obtain $4x + x\frac{dy}{dx} + y - 6y\frac{dy}{dx} = 1$.

We solve this equation for dy/dx :

$$x\frac{dy}{dx} - 6y\frac{dy}{dx} = 1 - 4x - y \Rightarrow (x - 6y)\frac{dy}{dx} = 1 - 4x - y \Rightarrow \frac{dy}{dx} = \frac{1-4x-y}{x-6y} \qquad ...(2)$$

BE CAREFUL : When applying Formula (2), keep in mind that the only values of x and y that can be substituted into the right-hand side of Formula (2) are those values that satisfy the original condition $2x^2 + xy - 3y^2 = x$. for instance, we might substitute x = 1, y = 2 to obtain (dy/dx) = (1 – 4 – 2)/(1 – 12) = 5/11; however, (1, 2) is not a point on the graph $2x^2 + xy - 3y^2 = x$, so the calculation of dy/dx at this point is totally meaningless.

The equation in Example can be written as $-3y^2 + xy + (2x^2 - x) = 0$ and hence is a quadratic equation in y (an equation of the form $Ay^2 + By + C = 0$ where A = –3, B = x and $C = 2x^2 - x$). Hence, we could use the quadratic formula to solve this equation for y in terms of x, obtaining

$$y = \frac{-x \pm \sqrt{25x^2 - 12x}}{-6}.$$

Though we are able to find y explicitly in terms of x, the resulting expression is fairly complex, and it still might be best to find dy/dx implicitly as in Example.

WARNING : It is important to realize that implicit differentiation is a technique for finding dy/dx that is valid only if y is a differentiable function of x, and careless application of the technique can lead to errors. For example, there is clearly no real-valued function y = f(x) that satisfies the equation $x^2 + y^2 = -1$, yet formal application of implicit differentiation yields the :derivative: dy/dx = –x/y. To be able to evaluate this "derivative," we must find some values for which $x^2 + y^2 = -1$. Because no such values exist, the derivative does not exist.

Ex.20 If $x^3 + y^2 = \sin(x + y)$, find $\frac{dy}{dx}$

Sol. given, $x^3 + y^3 = \sin(x + y)$ ...(i)

differentiating w. r. t. x, we get $\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}\{\sin(x + y)\}$

or $\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx}.\frac{dy}{dx} = \frac{d\sin(x+y)}{d(x+y)}.\frac{d(x+y)}{dx}$ or $3x^2 + 3y^2\frac{dy}{dx} = \cos(x + y).\left(1+\frac{dy}{dx}\right)$

or $[3y^2 - \cos(x + y)]\frac{dy}{dx} = \cos(x + y) - 3x^2$ $\therefore \frac{dy}{dx} = \frac{\cos(x+y)-3x^2}{3y^2-\cos(x+y)}$.

Ex.21 If $x = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y + \text{....to } \infty}}}$, prove that $\frac{dy}{dx} = 2x^2 + y^2 - 3xy$

Sol. $x = y + \cfrac{1}{y + \cfrac{1}{y + \cfrac{1}{y + \text{....to } \infty}}}$ $\therefore$ $x = y + \frac{1}{x}$ ...(i)

differentiating w. r. t. x, we get $1 = \frac{dy}{dx} - \frac{1}{x^2}$; or $\frac{dy}{dx} = 1 + \frac{1}{x^2}$

or $\frac{dy}{dx} = 1 + (x - y)^2$. $\left[\because \text{ from(i)}, \frac{1}{x} = x - y\right]$ $= 1 + x^2 + y^2 - 2xy$ ....(ii)

From (i) $x^2 = xy + 1$ $\therefore$ $1 = x^2 - xy$

Putting in (ii), we get $\frac{dy}{dx} = x^2 - xy + x^2 + y^2 - 2xy$ Hence $\frac{dy}{dx} = 2x^2 + y^2 - 3xy$

F. DERIVATIVE OF INVERSE FUNCTIONS

If the inverse functions f & g are defined by y = f(x) & x = g(y) & if

$f'(x)$ exists & $f'(x) \neq 0$ then $g'(y) = \frac{1}{f'(x)}$. This result can also be written as :

if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then $\frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right)$ or $\frac{dy}{dx}.\frac{dx}{dy} = 1$

We have $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$ We put u = x, $\frac{dy}{dx}.\frac{dx}{dy} = 1$.

The truth of this is also manifested geometrically, for $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are respectively the tangent and the co-tangent of the angle ψ which the tangent to the curve y = f(x) makes with the x-axis.

This formula is very useful in the differentiation of an inverse function.

Ex.22 Find the derivative of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and mention the points of non differentiability.

Sol. $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{bmatrix}$ $\Rightarrow$ $\frac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & \text{for} & |x| < 1 \\ \text{nonexistent} & \text{for} & |x| = 1 \\ -\frac{2}{1+x^2} & \text{for} & |x| > 1 \end{bmatrix}$

f is continuous for all x but not diff. at x = 1 , –1

G. DERIVATIVE OF A DETERMINANT

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Ex.23 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum of all the co-factors of the elements in f(0).

Sol. $f'(x) = \begin{vmatrix} 1 & 1 & 1 \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ 1 & 1 & 1 \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ 1 & 1 & 1 \end{vmatrix}$

$f''(x) = 0$ (obviously – two identical rows)

$f'(x) = k \;\Rightarrow\; f(x) = kx + x$, $f(0) = c \;\Rightarrow\; f(x) = f(0) + kx$. Note that $f'(x) = k$

$$\Rightarrow f'(0) = k = \begin{vmatrix} 1 & 1 & 1 \\ \ell & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ \ell & m & n \\ 1 & 1 & 1 \end{vmatrix}$$

$= (c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) + (c_{31} + c_{32} + c_{33})$ = sum of co-factors of elements f(0)]

H. HIGHER ORDER DERIVATIVES

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] & is called the first derivative of y w. r. t. x.

If it happens that the first derivative has a derivative on (a , b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Once we have found the derivative f′ of any function f, we can go on and find the derivative of f′.

$$f''(a) = \lim_{t\to 0} \frac{f'(a+t) - f'(a)}{t}.$$

Similarly, the 3rd order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$.

It is also denoted by $f'''(x)$ or y'''.

The **third derivative,** written f‴, is the derivative of the second derivative, and, in principle, we can go on forever and form derivatives of higher order. We adopt the alternative notation $f^{(n)}$ for the nth derivative of f.

Ex.24 Find y'' if $x^4 + y^4 = 16$.

Sol. Differentiating the equation implicitly with respect to x, we get $4x^3 + 4y^3y' = 0$

Solving for y′ gives $y' = -\frac{x^3}{y^3}$ ...(1)

to find y″ we differentiate this expression for y′ using the Quotient Rule and remembering that y is a function of x : $y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = \frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^2)}{(y^3)^2} = -\frac{y^3.3x^2 - x^3(3y^2y')}{y^6}$

If we now substitute Equation 1 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y} = -\frac{3(x^2y^4 + x^6)}{y^2} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

Ex.25 If y = x sin x, prove that $x^2 . \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + y)\, y = 0$

Sol. $\because$ y = x sin x ....(1) $\therefore$ $\frac{dy}{dx}$ = x cos x + sin x .....(2)

Again differentiating both sides w. r. t. x we get

$$\frac{d^2y}{dx^2} = x(-\sin x) + \cos x + \cos x = 2\cos x - x\sin x$$

or, $x^2 \frac{d^2y}{dx^2} = 2x^2 \cos x - x^2 . x \sin x = 2x^2 \cos x - x^2 y$ [from (1)]

$= 2x\left(\frac{dy}{dx} - \sin x\right) - x^2y$ [from (2)]

$= 2x\frac{dy}{dx} - 2x\sin x - x^2 y = 2x\frac{dy}{dx} - 2y - x^2y$ $\therefore$ $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + x^2)\, y = 0$

Ex.26 A function f(x) is so defined that for al x, $[f(x)]^n = f(nx)$. Prove that $f(x) . f'(nx) = f'(x) . f(nx)$, where $f'(x)$ denotes derivative of f(x) w.r. to x.

Sol. Given $[f(x)]^n = f(nx)$ .....(1)

differentiating both sides w. r to x, we get

$n[f(x)]^{n-1} . f'(x) = f'(nx) . (n . 1)$ or, $[f(x)^n . f'(x) = f'(nx)$

multiplying both sides by $f'(x)$, we get

$[f(x)^n . f'(x) = f'(nx) . f(x)$ or, $f(nx) . f'(x) = f'(nx) . f(x)$ [from (1)]

Ex.27 $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left(2\tan^{-1}\theta.\sqrt{\frac{1-x}{1+x}}\right)$, then prove that, $4(1 - x^2)^3 \left(\frac{d^2y}{dx^2}\right)^2 + 4x = x^2 + 4$.

Sol. y = u + v . Now $\frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}}$ & $v = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right) = \frac{2\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} = \sqrt{1-x^2}$ Hence

$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}}$ $\therefore$ $\frac{dy}{dx} = \frac{1-2x}{2\sqrt{1-x^2}}$

$\frac{d^2y}{dx^2} = \frac{x-2}{2(1-x)^{3/2}}$ (using quotient rule) $\Rightarrow$ $4(1 - x^2)^3 \frac{d^2y}{dx^2} = x^2 + y - 4x$

Ex.28 If $y = x^{n-1} . \ln x$ then use induction to prove that, $\dfrac{d^n y}{dx^n} = \dfrac{(n-1)!}{x}$ $\forall n \in N$.

Sol. $P(n) : D^n (x^{n-1} . \ln x) = \dfrac{(n-1)!}{x}$; $P(k+1) : D^{k+1} (x^k . \ln x) = \dfrac{k!}{x}$

L H S of $P(k+1) = D^k (x^{k-1} + \ln x . k\ x^{k-1})$ or $D^k (x^{k-1}) + k . D^k (\ln x . x^{k-1})$

$\Rightarrow 0 + k . \dfrac{(k-1)!}{x} = \dfrac{k!}{x}$ = R H S of P(k + 1)]

Ex.29 Change the independent variable to θ in the equation $\dfrac{d^2y}{dx^2} + \dfrac{2x}{1+x^2}\dfrac{dy}{dx} + \dfrac{y}{(1+x^2)^2} = 0$, by means of the transformation $x = \tan\theta$

Sol. We have $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \cos^2\theta \dfrac{dy}{d\theta}$; $\dfrac{dx}{d\theta} = \sec^2\theta \neq 0$ for any value of θ

Again $\dfrac{d^2y}{dx^2} = -2\cos\theta\sin\theta . \dfrac{d\theta}{dx}.\dfrac{dy}{d\theta} + \cos^2\theta . \dfrac{d^2y}{d\theta^2}\dfrac{d\theta}{dx}$

$= -2\cos\theta\sin\theta\cos^2\theta . \dfrac{dy}{d\theta} + \cos^2\theta . \dfrac{d^2y}{d\theta^2} . \cos^2\theta$

$= -2\sin\theta\cos^3\theta . \dfrac{dy}{d\theta} + \cos^4\theta \dfrac{d^2y}{d\theta^2}$.

Substituting the values of x, dy/dx and d^2y / dx^2 in the given differential equation, we have

$- 2\sin\theta\cos^3\theta \dfrac{dy}{d\theta} + \cos^4\theta \dfrac{d^2y}{d\theta^2} + \dfrac{2\tan\theta}{1+\tan^2\theta} . \cos^2\theta \dfrac{dy}{d\theta} + \dfrac{y}{(1+\tan^2\theta)^2} = 0$

$\Rightarrow - 2\sin\theta\cos^3\theta \dfrac{dy}{d\theta} + \cos^4\theta . \dfrac{d^2y}{d\theta^2} + 2\sin\theta . \cos^3\theta \dfrac{dy}{d\theta} + \cos^4\theta . y = 0 \Rightarrow \dfrac{d^2y}{d\theta^2} + y = 0.$

Ex.30 Given $F(x) = f(x) . \phi(x)$ and $f'(x) . \phi'(x) = c$ then prove that $\dfrac{F'''}{F} = \dfrac{f'''}{f} + \dfrac{\phi'''}{\phi}$, where $c \in$ constant, when $f(x), \phi(x), F(x)$ are differentiable

Sol. Given $F(x) = f(x) . \phi(x)$ ...(i) and $f'(x) . \phi'(x) = c$ ...(ii)

On differentiating (i) w.r.t. x, we get, $F'(x) = f(x) . \phi'(x) + f'(x) . \phi(x)$

Again differentiating ; $F''(x) = \{f(x)\,\phi''(x) + f'(x)\,\phi'(x)\} + \{f'(x) . \phi'(x) + f''(x)\,\phi(x)\}$

$F''(x) = f(x)\,\phi''(x) + 2f'(x)\,\phi'(x) + f''(x)\,\phi(x)$

or $F''(x) = f(x)\,\phi''(x) + f''(x)\,\phi(x) + 2c$ [using (ii)]

Now again differentiating w.r.t. x, we get

$F'''(x) = f(x) . \phi'''(x) + f'(x) . \phi''(x) + f''(x) . \phi'(x) + f'''(x) . \phi(x)$ ...(iii)

On differentiating (ii) w.r.t. x, we get $f'(x) . \phi''(x) + \phi'(x) . f''(x) = 0$ ...(iv)

from (iii) and (iv), we get, $F'''(x) = f(x)\,\phi'''(x) + \phi(x) . f'''(x) + 0$

or $\dfrac{F'''(x)}{F(x)} = \dfrac{f(x).\phi'''(x)}{F(x)} + \dfrac{\phi(x).f'''(x)}{F(x)} \Rightarrow \dfrac{F'''(x)}{F(x)} = \dfrac{\phi'''(x)}{\phi(x)} + \dfrac{f'''(x)}{f(x)}$ [$\because F(x) = f(x)\,\phi(x)$]

or $\dfrac{F'''}{F} = \dfrac{\phi'''}{\phi} + \dfrac{f'''}{f}$

I. L' HOPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces the indeterminate frorm 0/0, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of f(x)/g(x) as x approaches c produces any one of the indeterminate forms

∞ / ∞, $(-\infty) / \infty$, $\infty / (-\infty)$.

Ex.31 $\displaystyle \operatorname{Lim}_{\theta=0}\frac{e^{\theta}+e^{-\theta}+2\cos\theta-4}{\theta^4}$ $\left[\text{form } \frac{0}{0}\right]$ L' Hopital's Rule.

Sol. $\displaystyle = \operatorname{Lim}_{\theta=0}\frac{e^{\theta}-e^{-\theta}-2\sin\theta}{4\theta^3} = \operatorname{Lim}_{\theta=0}\frac{e^{\theta}+e^{-\theta}-2\cos\theta}{12\theta^2} = \operatorname{Lim}_{\theta=0}\frac{e^{\theta}-e^{-\theta}+2\sin\theta}{24\theta} = \operatorname{Lim}_{\theta=0}\frac{e^{\theta}+e^{-\theta}+2\cos\theta}{24} = \frac{4}{24} = \frac{1}{6}$.

Ex.32 Determine $\displaystyle \lim_{x\to 0}(\cot x)^{1/\log x}$

Sol. Let $y = (\cot x)^{1/\log x}$. $\displaystyle \log y = \frac{1}{\log x}\log(\cot x)$

$$\Rightarrow \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log\cot x}{\log x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to 0}\frac{\frac{-\operatorname{cosec}^2 x}{\cos x}}{1/x} = \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{\cos x} = -1 \quad \Rightarrow \quad \log \lim_{x\to 0} y = -1 \quad \Rightarrow \quad \lim_{x\to 0} y = e^{-1} = 1/e.$$

Ex.33 Find $\displaystyle \operatorname{Lim}_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$

Sol. The inconvenience of continuously differentiating the denominator, which involves $\tan^2 x$ as a factor, may be partially avoided as follows. We write

$$\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x} = \frac{1+\sin x-\cos x+\log(1-x)}{x^3}\cdot\left(\frac{x}{\tan x}\right)^2$$

so that $\displaystyle \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x} = \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\lim_{x\to 0}\left(\frac{x}{\tan x}\right)^2$

$$= \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\cdot 1 = \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$$

To evaluate the limit on the R.H.S., we notice that the numerator and denominator both become 0 for x = 0.

$$\therefore \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3} = \lim_{x\to 0}\frac{\cos x-\sin x-[1/(1-x)]}{3x^2}$$

$$= \lim_{x\to 0}\frac{-\sin x+\cos x-[1/(1-x)^2]}{6x} = \lim_{x\to 0}\frac{-\cos x-\sin x[2/(1-x)^3]}{6} = -\frac{3}{6} = -\frac{1}{2}$$

Ex.34 Find the constants 'a' (a > 0) and 'b' such that, $\underset{x\to 0}{\text{Limit}}\dfrac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx-\sin x} = 1.$

Sol. $\dfrac{0}{0}$ form hence using L` Hopital rule

$l = \underset{x\to 0}{\text{Limit}}\dfrac{\frac{x^2}{\sqrt{a+x}}}{b-\cos x}$ for existence of limit $\underset{x\to 0}{\text{Limit}}$ $b - \cos x = 0$ $\Rightarrow$ $b = 1$

hence $\underset{x\to 0}{\text{Limit}}\dfrac{x^2}{1-\cos x}\cdot\underset{x\to 0}{\text{Limit}}\dfrac{1}{\sqrt{a+x}} = 1$ $\quad \dfrac{2}{\sqrt{a}} = 1$ $\Rightarrow$ $a = 4$

Ex.35 Evaluate $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}+\sqrt{x}+x\sqrt{x}-3}{x^3-1}$

Sol. $\lim\limits_{x\to 1}\dfrac{(\sqrt[3]{x}-1)+(\sqrt{x}-1)+(x^{3/2}-1)}{(x-1)(x^2+x+1)}$ $\quad\left(\dfrac{0}{0}\text{form}\right)$

$= \lim\limits_{x\to 1}\left\{\dfrac{\sqrt[3]{x}-1}{x-1}+\dfrac{(\sqrt{x}-1)}{x-1}+\dfrac{(x^{3/2}-1)}{x-1}\right\}\cdot\dfrac{1}{x^2+x+1}$

$= \left\{\left(\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x-1}\right)+\left(\lim\limits_{x\to 1}\dfrac{(\sqrt{x}-1)}{x-1}\right)+\left(\lim\limits_{x\to 1}\dfrac{(x^{3/2}-1)}{x-1}\right)\right\}\cdot\lim\limits_{x\to 1}\dfrac{1}{x^2+x+1}$

(Apply L–Hopital's rule)

$= \left\{\dfrac{1}{3}(1)^{1/3-1}+\dfrac{1}{2}(1)^{1/2-1}+\dfrac{3}{2}(1)^{3/2-1}\right\}.\dfrac{1}{1^2+1+1}$

$= \left\{\dfrac{1}{3}(1)^{-2/3}+\dfrac{1}{2}(1)^{-1/2}+\dfrac{3}{2}(1)^{1/2}\right\}.\dfrac{1}{3}$ $\quad = \left(\dfrac{1}{3}+\dfrac{1}{2}+\dfrac{3}{2}\right)\cdot\dfrac{1}{3}$ $\quad = \left(\dfrac{2+3+9}{6}\right)\cdot\dfrac{1}{3}=\dfrac{7}{9}$

Ex.36 Evaluate : $\lim\limits_{x\to 0}\left(\dfrac{1}{x^5}\int_0^x e^{-t^2}dt-\dfrac{1}{x^4}+\dfrac{1}{3x^2}\right)$

Sol. We have $\lim\limits_{x\to 0}\dfrac{3\int_0^x e^{-t^2}dt-3x+x^3}{x^5}$ $\quad\left(\text{form}\dfrac{0}{0}\right)$

Using Hopital's rule, we get $\quad \lim\limits_{x\to 0}\dfrac{3e^{-x^2}-3+3x^2}{5x^4}$ $\quad \lim\limits_{x\to 0}\dfrac{e^{-x^2}-1+x^2}{x^4}$

(Applying Newton–Leibnitz's formula to $\dfrac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}.\dfrac{d}{dx}(x)-e^{-0}\dfrac{d}{dx}(0) = e^{-x^2}$)

Again using Hopital's rule, we get $\quad = \dfrac{3}{5}\lim\limits_{x\to 0}\dfrac{e^{-x^2}.(-2x)-0+2x}{4x^3} = \dfrac{3}{5}\cdot\dfrac{2}{4}\lim\limits_{x\to 0}\dfrac{e^{-x^2}+1}{x^2}$

Again using Hopital's rule, we get $\quad = \dfrac{3}{10}\lim\limits_{x\to 0}\dfrac{e^{-x^2}.(-2x)}{(2x)} = \dfrac{3}{10}\cdot 1 = \dfrac{3}{10}$

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. If y = f(x) is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(A) 4 (B) 2 (C) – 2 (D) 0

Sol.

2. If $f(x) = \log_x (\ln x)$ then $f'(x)$ at x = e is

(A) 1/e (B) e (C) 1 (D) zero

Sol.

3 . If $y = \cos^{-1}(\cos x)$ then $\frac{dy}{dt}$ at $x = \frac{5\pi}{4}$ is equal to

(A) 1 (B) – 1 (C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Sol.

4. If $x = \frac{1+t}{t^3}$, $y = \frac{3}{2t^2} + \frac{2}{t}$ then, $x\left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} =$

(A) 0 (B) – 1 (C) 1 (D) 2

Sol.

5. If sin (xy) + cos (xy) = 0 then $\frac{dy}{dx} =$

(A) $\frac{y}{x}$ (B) $-\frac{y}{x}$ (C) $-\frac{x}{y}$ (D) $\frac{x}{y}$

Sol.

6. If $y = x^{x^2}$ then $\frac{dy}{dx} =$

(A) $2 \ln x \cdot x^{x^2}$ (B) $(2 \ln x + 1) \cdot x^{x^2}$

(C) $(2 \ln x + 1) \cdot x^{x^2+1}$ (D) none of these

Sol.

7. If $f(x) = |x|^{|\sin x|}$ then $f'(\pi/4)$ equals

(A) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right)$

(B) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}\right)$

(C) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4} - \frac{2\sqrt{2}}{\pi}\right)$

(D) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4} + \frac{2\sqrt{2}}{\pi}\right)$

Sol.

8. If $y = \sin^{-1}\dfrac{x^2-1}{x^2+1} + \sec^{-1}\dfrac{x^2+1}{x^2-1}$, $|x| > 1$ then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{x}{x^4-1}$ (B) $\dfrac{x^2}{x^4-1}$ (C) 0 (D) 1

Sol.

9. If $y = x - x^2$, then the derivative of y^2 w.r.t. x^2 is
(A) $2x^2 + 3x - 1$ (B) $2x^2 - 3x + 1$
(C) $2x^2 + 3x + 1$ (D) none of these
Sol.

10. Let f(x) be a polynomial in x. Then the second derivative of $f(e^x)$, is
(A) $f''(e^x) \cdot e^x + f'(e^x)$ (B) $f''(e^x) \cdot e^{2x} + f'(e^x).e^{2x}$
(C) $f''(e^x)\, e^{2x}$ (D) $f''(e^x) \cdot e^{2x} + f'(e^x) \cdot e^x$
Sol.

11. If $x = at^2$, $y = 2at$, then $\dfrac{d^2y}{dx^2}$ is

(A) $-\dfrac{1}{t^2}$ (B) $\dfrac{1}{2at^2}$ (C) $-\dfrac{1}{t^3}$ (D) $-\dfrac{1}{2at^3}$

Sol.

12. If f(x), g(x), h(x) are polynomials in x of degree 2 and $F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$, then $F'(x)$ is equal to

(A) 1 (B) 0
(C) –1 (D) f(x) . g(x) . h(x)
Sol.

13. If $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$

and $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x(1-x)}} + p$, then p =

(A) 0 (B) $\dfrac{1}{\sqrt{1-x}}$ (C) $\sin^{-1}\sqrt{x}$ (D) $\dfrac{1}{\sqrt{1-x^2}}$

Sol.

14. If $\dfrac{d}{dx}\left(\dfrac{1+x^2+x^4}{1+x+x^2}\right) = ax + b$ then the value of a and b are respectively
(A) 2 and 1 (B) –2 and 1
(C) 2 and –1 (D) none of these
Sol.

15. Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. Then $\underset{x\to 0}{\text{Limit}}\dfrac{f'(x)}{x} =$

(A) 2 (B) –2 (C) –1 (D) 0

Sol.

16. If u = ax + b then $\frac{d^n}{dx^n}$ (f(ax + b)) is equal to

(A) $\frac{d^n}{du^n}$ (f(u)) (B) $a\frac{d^n}{du^n}$ (f(u))

(C) $a^n \frac{d^n}{du^n}$ (f(u)) (D) $a^{-n} \frac{d^n}{du^n}$ (f(u))

Sol.

17. If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is

(A) e^x (B) $\frac{-e^x}{(1+e^x)^3}$ (C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{e^x}{(1+e^x)^2}$

Sol.

18. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x$ then $\frac{dy}{dx}$ =

(A) $\frac{1+x-x^2}{(1+x^2)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$ (B) $\frac{2(1+x-x^2)}{(1+x^2)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$

(C) $\frac{1-x+x^2}{(1+x^2)^2}\sin\left(\frac{2x-1}{x^2+1}\right)$ (D) none of these

Sol.

19. If $8 f(x) + 6 f\left(\frac{1}{x}\right) = x + 5$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at x = −1 is equal to

(A) 0 (B) $\frac{1}{14}$ (C) $-\frac{1}{14}$ (D) none of these

Sol.

20. If $x = e^{y+e^{y+....to\,\infty}}$, x > 0, then $\frac{dy}{dx}$

(A) $\frac{x}{1+x}$ (B) $\frac{1}{x}$ (C) $\frac{1-x}{x}$ (D) $\frac{1+x}{x}$

Sol.

21. If $f(x) = x^n$, then the value of

$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + ... + \frac{(-1)^n f^n(1)}{n!}$ is

(A) 2^n (B) 2^{n-1} (C) 0 (D) 1

Sol.

22. If $y = \frac{a + bx^{3/2}}{x^{5/4}}$ & $\frac{dy}{dx}$ vanishes when x = 5 then $\frac{a}{b}$ =

(A) $\sqrt{3}$ (B) 2 (C) $\sqrt{5}$ (D) None of these

Sol.

23. If f(x) = f′(x) + f″(x) + f‴(x) + f⁗(x) ....∞ also f(0) = 1 and f(x) is a differentiable function indefinitely then f(x) has the value

(A) e^x (B) $e^{x/2}$ (C) e^{2x} (D) e^{4x}

Sol.

24. If $y = \sin^{-1} \frac{2x}{1+x^2}$ then $\left.\frac{dy}{dx}\right]_{x=-2}$ is

(A) $\frac{2}{5}$ (B) $\frac{2}{\sqrt{5}}$ (C) $-\frac{2}{5}$ (D) None of these

Sol.

25. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ =

(A) $\frac{\sin x}{2y-1}$ (B) $\frac{\sin x}{1-2y}$ (C) $\frac{\cos x}{1-2y}$ (D) $\frac{\cos x}{2y-1}$

Sol.

26. If $y = e^{-x} \cos x$ and $y_4 + ky = 0$, where $y_4 = \frac{d^4y}{dx^4}$, then k =

(A) 4 (B) −4 (C) 2 (D) −2

Sol.

27. If y = a cos (*l*n x) + b sin (*l*n x), then $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx}$

(A) 0 (B) y (C) −y (D) None of these

Sol.

28. If y = sin mx then the value of $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ (where subscripts of y shows the order of derivative) is

(A) independent of x but dependent on m

(B) dependent of x but independent of m

(C) dependent on both m & x

(D) independent of m & x

Sol.

29. If f is differentiable in (0, 6) & $f'(4) = 5$ then $\lim\limits_{x\to 2}\dfrac{f(4)-f(x^2)}{2-x}$ =

(A) 5 (B) 5/4 (C) 10 (D) 20

Sol.

30. Let $y = e^{2x}$. Then $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{d^2x}{dy^2}\right)$ is

(A) 1 (B) e^{-2x} (C) $2e^{-2x}$ (D) $-2e^{-2x}$

Sol.

31. If g is the inverse function of f an $f'(x) = \dfrac{x^5}{1+x^4}$.

If g(2) = a, then f′(2) is equal to

(A) $\dfrac{a^5}{1+a^4}$ (B) $\dfrac{1+a^4}{a^5}$ (C) $\dfrac{1+a^5}{a^4}$ (D) $\dfrac{a^4}{1+a^5}$

Sol.

32. If $y = (1 + x)(1 + x^2)(1 + x^4) \ldots\ldots (1 + x^{2n})$, then $\dfrac{dy}{dx}$ at x = 0 is

(A) −1 (B) 1 (C) 0 (D) 2^n

Sol.

33. The derivative of $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$

at $x = \dfrac{1}{2}$ is

(A) 4 (B) 1/4 (C) 1 (D) None of these

Sol.

34. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\dfrac{\pi}{2}\right)$ =

(A) 0 (B) 1 (C) 4 (D) None of these

Sol.

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. The differential coefficient of $\sin^{-1}\frac{t}{\sqrt{1+t^2}}$ w.r.t $\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ is

(A) $1 \ \forall\ t > 0$ (B) $-1 \ \forall\ t < 0$
(C) $1 \ \forall\ t \in R$ (D) none of these

Sol.

2. If $f(x) = |(x - 4)(x - 5)|$, then $f'(x)$ is
(A) $-2x + 9$, for all $x \in R$ (B) $2x - 9$ if $x > 5$
(C) $-2x + 9$ if $4 < x < 5$ (D) not defined for $x = 4, 5$

Sol.

3. If $x^p \cdot y^q = (x + y)^{p+q}$ then $\frac{dy}{dx}$ is

(A) independent of p (B) independent of q
(C) dependent on both p and q (D) $\frac{y}{x}$

Sol.

4. The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation

(A) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ (B) $\frac{d^2u}{dx^2} = 2v$

(C) $\frac{d^2v}{dx^2} = -2u$ (D) $\frac{du}{dx} + \frac{dv}{dx} = 2v$

Sol.

5. If $\sqrt{x^2 + y^2} = e^t$ where $t = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$ then $\frac{dy}{dx} =$

(A) $\frac{x-y}{x+y}$ (B) $\frac{x+y}{x-y}$ (C) $\frac{y-x}{y+x}$ (D) $\frac{x-y}{2x+y}$

Sol.

6. If $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$, then $\frac{d}{dx}\{f_n(x)\}$ is equal to

(A) $f_n(x) \cdot \frac{d}{dx}\{f_{n-1}(x)\}$ (B) $f_n(x) \cdot f_{n-1}(x)$
(C) $f_n(x) \cdot f_{n-1}(x) \ldots\ldots f_2(x) \cdot f_1(x)$
(D) none of these

Sol.

7. If f is twice differentiable such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x)$ is twice differentiable function such that $h'(x) = [f(x)]^2 + [g(x)]^2$. If $h(0) = 2$, $h(1) = 4$, then the equation $y = h(x)$ represents
(A) a curve of degree 2
(B) a curve passing through the origin
(C) a straight line with slope 2
(D) a straight line with y intercept equal to 2.
Sol.

8. If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ then
(A) $f(-2) = 0$ (B) $f'(-1/2) = 0$
(C) $f'(-1) = 2$ (D) $f''(0) = 4$
Sol.

9. If $f(x) = (ax + b) \sin x + (cx + d) \cos x$, then the values of a, b, c and d such that $f'(x) = x \cos x$ for all x are
(A) a = d = 1 (B) b = 0 (C) c = 0 (D) b = c
Sol.

10. $y = \cos^{-1}\sqrt{\dfrac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$ then $\dfrac{dy}{dx}$ is
(A) $\dfrac{1}{2(1+x^2)}$, $x \in R$ (B) $\dfrac{1}{2(1+x^2)}$, $x > 0$
(C) $\dfrac{-1}{2(1+x^2)}$, $x < 0$ (D) $\dfrac{1}{2(1+x^2)} < 0$
Sol.

11. Two functions f & g have first & second derivatives at x = 0 satisfy the relations, $f(0) = \dfrac{2}{g(0)}$, $f'(0) = 2$ $g'(0) = 4g(0)$, $g''(0) = 5 f''(0) = g(0) = 3$ then
(A) if $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(0) = \dfrac{15}{4}$
(B) if $k(x) = f(x) \,.\, g(x) \sin x$ then $k'(0) = 2$
(C) $\lim\limits_{x\to 0}\dfrac{g'(x)}{f'(x)} = \dfrac{1}{2}$ (D) None of these
Sol.

12. If $y = \tan^{-1}\left(\dfrac{\ln \dfrac{e}{x^2}}{\ln ex^2}\right) + \tan^{-1}\dfrac{3+2\ln x}{1-6\ln x}$ then
(A) $\dfrac{dy}{dx} = 0$ (B) $\dfrac{d^2y}{dx^2} = 0$
(C) $\dfrac{dy}{dx} = \dfrac{2}{x(1+\ln^2 x)}$ (D) $\dfrac{dy}{dx} = 1$
Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. Find the derivative of following functions with respect to x from the first principle (ab – initio method).
(i) $f(x) = \sin x^2$
Sol.

(ii) $f(x) = e^{2x+3}$
Sol.

2. Differentiate the following functions with respect to x.

(i) $x^{2/3} + 7e - \frac{5}{x} + 7\tan x$

Sol.

(ii) $x^2 \,.\, \ln x \,.\, e^x$
Sol.

(iii) $\ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

Sol.

(iv) $\frac{\sin x - x\cos x}{x\sin x + \cos x}$

Sol.

(v) $\tan\left(\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Sol.

3. If $f(x) = 2\ ln(x - 2) - x^2 + 4x + 1$, then find the solution set of the inequality $f'(x) \geq 0$.
Sol.

4. Find $\frac{dy}{dx}$ when x and y are connected by the following relations
(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
Sol.

(ii) $xy + xe^{-y} + y \,.\, e^x = x^2$
Sol.

5. Differentiate the given functions w.r.t.x.
(i) $(ln\ x)^{\cos x}$
Sol.

(ii) $x^x - 2^{\sin x}$

Sol.

(iii) $y = (x\, ln\, x)^{ln\, ln\, x}$

Sol.

6. If P_n is the sum of GP upon n terms. Show that

$(1 - r)\dfrac{dP_n}{dr} = n \,.\, P_{n-1} - (n-1)\, P_n$.

Sol.

7. If $x = a\, t^3$ and $y = b\, t^2$, where t is a parameter, then prove that $\dfrac{d^3y}{dx^3} = \dfrac{8b}{27a^3.t^7}$

Sol.

8. Show that the substitution $z = ln\left(\tan\dfrac{x}{2}\right)$ changes the equation $\dfrac{d^2y}{dx^2} + \cot x\,\dfrac{dy}{dx} + 4y\,\text{cosec}^2 x = 0$ to $(d^2y/dz^2) + 4y = 0$.

Sol.

9. If $f(x) = x^n$ then find the value of

$f(1) + \dfrac{f^1(1)}{1!} + \dfrac{f^2(1)}{2!} + \ldots\ldots + \dfrac{f^n(1)}{n!}$ where $f^r(x)$ denotes the r^{th} derivative of f(x) w.r.t. x

Sol.

10. If $\underset{x\to 0}{\text{Limit}}\dfrac{a\sin x - bx + cx^2 + x^3}{2x^2.\ln(1+x) - 2x^3 + x^4}$ exists and is finite, find the values of a, b, c and the limit.

Sol.

11. If $\cos\dfrac{x}{2}.\cos\dfrac{x}{2^2}.\cos\dfrac{x}{2^3}\ldots\infty = \dfrac{\sin x}{x}$ then find the value of $\dfrac{1}{2^2}\sec^2\dfrac{x}{2} + \dfrac{1}{2^4}\sec^2\dfrac{x}{2^2} + \dfrac{1}{2^6}\sec^2\dfrac{x}{2^3}\ldots\infty$.

Sol.

12. Show that the function y = f(x) defined by the parametric equations $x = e^t \sin t$, $y = e^t \cos t$ satisfies the relation $y''(x + y)^2 = 2\,(xy' - y)$.

Sol.

13. If y = x log $\left(\frac{x}{a+bx}\right)$, then prove that $x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

Sol.

14. If y = $(\cos x)^{\ln x} + (\ln x)^x$ find $\frac{dy}{dx}$.

Sol.

15. Suppose f(x) = tan $(\sin^{-1}(2x))$

(a) Find the domain and range of f.

Sol.

(b) Express f(x) as an algebraic function of x.

Sol.

(c) Find f′(1/4).

Sol.

16. Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + ...\infty}}}$. Compute the value of f(100) . f′(100).

Sol.

17. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w.r.t. $\sqrt{1-x^4}$.

Sol.

18. Find the derivative with respect to x of the function

$(\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \arcsin\frac{2x}{1+x^2}$ at $x = \frac{\pi}{4}$

Sol.

19. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$, prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

Sol.

20. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots}}}$, prove that

$$\frac{dy}{dx} = \cfrac{1}{2 - \cfrac{x}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots\ldots\ldots}}}}.$$

Sol.

21. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$,

$u \in \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$ prove that $2\frac{dy}{dx} + 1 = 0$.

Sol.

22. If $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$, find $\frac{dy}{dx}$

if $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

Sol.

23. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$,

then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

Sol.

24. (a) Let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. Find the value of g' where $f(x) = 2$.

Sol.

(b) Let f, g and h are differentiable functions.
If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ and the derivatives of their pair wise products at $x = 0$ are $(fg)'(0) = 6$; $(gh)'(0) = 4$ and $(hf)'(0) = 5$ then compute the value of $(fgh)'(0)$.

Sol.

25. If x = 2 cost – cos2t & y = 2sint – sin2t, find the value of $\left(\frac{d^2y}{dx^2}\right)$ when t = $\left(\frac{\pi}{2}\right)$.

Sol.

26. If f : R $\to$ R is a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$, then prove that f(2) = f(1) – f(0).

Sol.

27. If $y = x\, \ln [(ax)^{-1} + a^{-1}]$, prove that

$$x(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y - 1.$$

Sol.

28. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$.

Find the continuous function f(x) satisfying $f'(1) = f(-1)$

Sol.

29. If sin y = x sin (a + y), show that

$$\frac{dy}{dx} = \frac{\sin a}{1 - 2x\cos a + x^2}.$$

Sol.

30. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} + \ldots\ldots$ to n terms. Find dy/dx, expressing your answer in 2 terms.

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. Prove that
if $| a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx | \le | \sin x |$ for $x \in R$, then $|a_1 + 2a_2 + 3a_3 + ... + na_n| \le 1$
Sol.

2. The function $f : R \to R$ satisfies
$f(x^2) . f''(x) = f'(x) . f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.
Sol.

3. Let $y = x \sin kx$. Find the possible vale of k for which the differential equation $\frac{d^2y}{dx^2} + y = 2k \cos kx$ holds true for all $x \in R$.
Sol.

4. Let $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$. Define the function $f'(x)$ for all x find $f''(0)$ if it exist.
Sol.

5. Show that the substitution $z = \ln\left(\tan\frac{x}{2}\right)$ changes the equation $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4 y \operatorname{cosec}^2 x = 0$ to $\frac{d^2y}{dz^2} + 4y = 0$.
Sol.

6. Prove that $\cos x + \cos 3x + \cos 5x + ... + \cos (2n-1) x = \frac{\sin 2nx}{2 \sin x}$, $x \neq K\pi$, $K \in I$ and deduce from this:
$\sin x + 3 \sin 3x + 5 \sin 5x + ... + (2n - 1) \sin (2n - 1) x = \frac{[(2n+1)\sin(2n-1)x - (2n-1)\sin(2n+1)x]}{4\sin^2 x}$.
Sol.

7. Find a polynomial function f(x) such that $f(2x) = f'(x) f''(x)$.
Sol.

8. (i) Let $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x < 0 \end{cases}$ then prove that
(a) f is continuous and differentiable for all x.
Sol.

(b) f' is continuous and differentiable for all x.

Sol.

(ii) $f : [0, 1] \to R$ is defined as

$$f(x) = \begin{bmatrix} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if} & 0 < x \le 1 \\ 0 & \text{if} & x = 0 \end{bmatrix}$$, then prove that

(a) f is differentiable in [0, 1]

Sol.

(b) f is bounded in [0, 1]

Sol.

(c) f is bounded in [0, 1]

Sol.

9. Let f(x) be a derivable function at x = 0 & $f\left(\frac{x+y}{k}\right) = \frac{f(x)+f(y)}{k}$ ($k \in R, k \neq 0, 2$). Show that f(x) is either a zero or an odd linear function.

Sol.

10. Let $\frac{f(x)-f(y)}{2} = \frac{f(y)-a}{2}$ + xy for all real x and y. If f(x) is differentiable and $f'(0)$ exists for all real permissible values of 'a' and is equal to $\sqrt{5a-1-a^2}$. Prove that f(x) is positive for all real x.

Sol.

11. If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then

$f'(x) = \lambda \cdot \begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$. Find the value of λ.

Sol.

12. Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum of all the co–factors of the elements in f(0).

Sol.

13. $\lim\limits_{x\to 0}\left[\frac{1}{x\sin^{-1}x} - \frac{1-x^2}{x^2}\right]$

Sol.

14. $\lim\limits_{x\to 0}\frac{x\cos x - \ln(1+x)}{x^2}$

Sol.

15. If $\lim\limits_{x\to a}\dfrac{a^x - x^a}{x^x - a^a}$ find 'a'.

Sol.

16. $\lim\limits_{x\to 0}\dfrac{1+\sin x - \cos x + \ln(1-x)}{x.\tan^2 x}$

Sol.

17. Determine the values of a, b and c so that $\lim\limits_{x\to 0}\dfrac{(a+b\cos x)x - c\sin x}{x^5} = 1$.

Sol.

18. $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sin x - (\sin x)^{\sin x}}{1-\sin x + \ln(\sin x)}$

Sol.

19. $\lim\limits_{x\to 0}\dfrac{3x\ln\left(\dfrac{\sin x}{x}\right)^2 + x^3}{(x-\sin x)(1-\cos x)}$

Sol.

20. Find the value of f(0) so that the function

$f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$, $x \neq 0$ is continuous at x = 0 & examine the differentiability of f(x) at x = 0.

Sol.

21. If $\lim\limits_{x\to 0}\dfrac{a\sin x - bx + cx^2 + x^3}{2x^2.\ln(1+x) - 2x^3 + x^4}$ exists & is finite, find the vales of a, b, c & the limit.

Sol.

22. Evaluate : $\lim\limits_{x\to 0}\dfrac{x^{6000} - (\sin x)^{6000}}{x^2.(\sin x)^{6000}}$

Sol.

23. If $\lim\limits_{x\to 0}\dfrac{1-\cos x.\cos 2x.\cos 3x.....\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n \in N$)

Sol.

EXERCISE – V

JEE PROBLEMS

1. If $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$, then find the domain and the range of f. Show that f is one-one. Also find the function $\dfrac{df^{-1}(x)}{dx}$ and its domain. **[REE 99,6]**

Sol.

2. (a) If $x^2 + y^2 = 1$ then **[JEE 2000 (Scr.), 1]**

(A) $yy'' - 2(y')^2 + 1 = 0$ (B) $yy'' + (y')^2 + 1 = 0$

(C) $yy'' - (y')^2 - 1 = 0$ (D) $yy'' + 2(y')^2 + 1 = 0$

Sol.

(b) Suppose $p(x) = a_0 + a_1 x + a_2x^2 + ... + a_nx^n$. If $|\ p(x)\ | \le |\ e^{x-1} - 1|$ for all $x \ge 0$ prove that $|a_1 + 2\,a_2 + ... + n\,a_n| \le 1$. **[JEE 2000 (Mains), 5]**

Sol.

3. (a) If $\ell n\,(x + y) = 2xy$, then $y'(0) =$ **[JEE 2004 (Scr.)]**

(A) 1 (B) –1 (C) 2 (D) 0

Sol.

(b) $f(x)=\begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2} & \text{at } x = 0 \\ \dfrac{e^{ax/2}-1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$ **[JEE 2004, 4]**

If f(x) is differentiable at x = 0 and $|\ c\ | < 1/2$ then find the value of 'a' and prove that $64b^2 = 4 - c^2$.

Sol.

4. (a) If y = y(x) and it follows the relation $x \cos y + y \cos x = \pi$, then $y''(0)$ **[JEE 2005 (Scr.)]**

(A) 1 (B) –1 (C) π (D) $-\pi$

Sol.

(b) If P(x) is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(1) = 1$, $P(0) = 0$ and $P'(x) > 0\ \forall\ x \in [0, 1]$, then

(A) $S = \phi$

(B) $S = \{(1 - a)x^2 + ax,\ 0 < a < 2\}$

(C) $(1 - a)x^2 + ax,\ a \in (0, \infty)$

(D) $S = \{(1 - a)x^2 + ax,\ 0 < a < 1\}$

Sol.

(c) If f(x) is a continuous and differentiable function and $f(1/n) = 0,\ \forall\ n \ge 1$ and $n \in I$, then

(A) $f(x) = 0,\ x \in (0, 1]$

(B) $f(0) = 0,\ f'(0) = 0$

(C) $f'(x) = 0 = f''(x),\ x \in (0, 1]$

(D) $f(0) = 0$ and $f'(0)$ need not to be zero

Sol.

(d) If $f(x - y) = f(x) \cdot g(y) - f(y) \cdot g(x)$ and $g(x - y) = g(x) \cdot g(y) + f(x) \cdot f(y)$ for all $x, y \in R$. If right hand derivative at x = 0 exists for f(x). Find derivative of g(x) and x = 0. **[JEE 2005 (Mains), 4]**

Sol.

5. For x > 0, $\lim\limits_{x\to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is **[JEE 2006, 3]**

(A) 0 (B) –1 (C) 1 (D) 2

Sol.

6. $\dfrac{d^2x}{dy^2}$ equals **[JEE 2007, 3]**

(A) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

(C) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (D) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Sol.

7. (a) Let g(x) = ℓn f(x) where f(x) is a twice differentiable positive function on (0, ∞) such that f(x + 1) = x f(x). Then for N = 1, 2, 3 ;

$g''\left(N+\dfrac{1}{2}\right) - g''\left(\dfrac{1}{2}\right) =$ **[JEE 2008, 3 + 3]**

(A) $-4\left\{1+\dfrac{1}{9}+\dfrac{1}{25}+.....+\dfrac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\dfrac{1}{9}+\dfrac{1}{25}+.....+\dfrac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\dfrac{1}{9}+\dfrac{1}{25}+.....+\dfrac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\dfrac{1}{9}+\dfrac{1}{25}+.....+\dfrac{1}{(2N+1)^2}\right\}$

Sol.

(b) Let f and g be real valued functions defined on interval (–1, 1) such that g″(x) is continuous, g(0) ≠ 0, g′(0) = 0, g″(0) ≠ 0, and f(x) = g(x) sin x.

Statement–1 : $\lim\limits_{x\to 0}$ [g(x) cot x – g(0) cosec x] = f″(0).

and

Statement–2 : f′(0) = g(0).

(A) Statement (1) is correct and statement (2) is correct and statement (2) is correct explanation for (1)

(B) Statement (1) is correct and statement (2) is correct and statement (2) is NOT correct explanation for (1)

(C) Statement (1) is true but (2) is false

(D) Statement (1) is false but (2) is true

Sol.

8. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of g′(1) is **[JEE 2009]**

Sol.

9. Let $f(\theta)=\sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\dfrac{\pi}{4}<\theta<\dfrac{\pi}{4}$.

Then the value of $\dfrac{d}{d(\tan\theta)}$ (f(θ)) is **[JEE 2011]**

Sol.

10. Let y'(x) + y(x)g'(x) = g(x)g'(x), y(0) = 0, x ∈ R, where f'(x) denotes $\dfrac{d\,f(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is **[JEE 2011]**

Sol.

Answer Ex–I — SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. C **2.** A **3.** B **4.** C **5.** B **6.** C **7.** D **8.** C
9. B **10.** D **11.** D **12.** B **13.** D **14.** C **15.** B **16.** C
17. B **18.** B **19.** C **20.** C **21.** C **22.** C **23.** B **24.** C
25. D **26.** A **27.** C **28.** D **29.** D **30.** D **31.** B **32.** B
33. A **34.** C

Answer Ex–II — MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. AB **2.** BCD **3.** ABD **4.** ABCD **5.** BC **6.** AC **7.** CD
8. BCD **9.** ABCD **10.** BC **11.** ABC **12.** AB

Answer Ex–III — SUBJECTIVE QUESTIONS

1. (i) $f'(x) = 2x \cos x^2$ **(ii)** $f'(x) = 2\, e^{2x+3}$

2. (i) $\frac{2}{3x^{\frac{1}{3}}} + \frac{5}{x^2} + 7 \sec^2 x$ **(ii)** $e^x\, x\, (2\, \ln x + 1 + x\, (\ln x)$ **(iii)** $\sec x$

(iv) $\frac{x^2}{(x \sin x + \cos x)^2}$ **(v)** $\frac{1}{2} \sec^2 \frac{x}{2}$

3. (2, 3] **4. (i)** $-\frac{ax + hy + g}{hx + by + f}$ **(ii)** $\frac{2x - y - e^{-y} - e^x y}{x - xe^{-y} + e^x}$

5. (i) $(\ln x)^{\cos x} \left(\frac{\cos x}{x \ln x} - \sin x \ln(\ln x)\right)$ **(ii)** $x^x\,(1 + \ln x) - \ln 2 \,.\, 2^{\sin x} \,.\, \cos x$

(iii) $(x \ln x)^{\ln \ln x}\ \frac{1}{x}\left(1 + \ln(\ln x)\left(1 + \frac{2}{\ln x}\right)\right)$ **9.** 2^n **10.** $a = 6,\ b = 6,\ c = 0;\ \frac{3}{40}$

11. $\text{cosec}^2\, x - (1/x^2)$ **14.** $Dy = (\cos x)^{\ln x} \left[\frac{\ln(\cos x)}{x} - \tan x \ln x\right] + (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x)\right]$

15. (a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$ **(b)** $f(x) = \frac{2x}{\sqrt{1 - 4x^2}}$ **(c)** $\frac{16\sqrt{3}}{9}$ **16.** 100

17. $\frac{1 + \sqrt{1 + x^4}}{x^6}$ **18.** $\frac{32}{16 + \pi^2} - \frac{8}{\ln 2}$ **22.** $\frac{1}{2}$ or $-\frac{1}{2}$

23. $\frac{1 - 2x}{2\sqrt{1 - x^2}}$ **24. (a)** 1/6 ; **(b)** 16 **25.** $-\frac{3}{2}$

28. $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ln\frac{3}{2}\right]x & \text{if} \quad x \le 0 \\ \left(\frac{1 + x}{2 + x}\right)^{1/x} & \text{if} \quad x > 0 \end{cases}$ **30.** $\frac{1}{1 + (x + n)^2} - \frac{1}{1 + x^2}$

Answer Ex–IV — ADVANCED SUBJECTIVE QUESTIONS

2. 6 **3.** k = 1, –1 or 0 **4.** $f'(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\dfrac{1}{3}$

7. $\dfrac{4x^3}{9}$ **11.** 3 **13.** $\dfrac{5}{6}$ **14.** $\dfrac{1}{2}$

15. a = 1 **16.** $-\dfrac{1}{2}$ **17.** a = 120; b = 60; c = 180 **18.** 2

19. –2/5 **20.** f(0) = 1; differentiable at x = 0, $f'(0^+) = -(1/3)$; $f'(0^-) = -(1/3)$

21. a = 6, b = 6, c = 0; $\dfrac{3}{40}$ **22.** 1000 **23.** n = 11

Answer Ex–V — JEE PROBLEMS

1. Domain of f(x) = R – {–2, 0} ; Range of f(x) = R – {–1/2, 1} ; $\dfrac{d}{dx}[f^{-1}(x)] = \dfrac{3}{(1-x)^2}$

Domain of $f^{-1}(x)$ = R – {– 1/2, 1} **2. (a)** B **3. (a)** A ; **(b)** a = 1

4. (a) C ; **(b)** B ; **(c)** B, **(d)** $g'(0) = 0$ **5.** C **6.** D **7. (a)** A, **(b)** A **8.** 2

9. 1 **10.** 0